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Instability due to viscosity stratification

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The principal aim of this paper is to show that the variation of viscosity in a fluid can cause instability. Plane Couette–Poiseuille flow of two superposed layers of fluids of different viscosities between two horizontal plates is considered, and it is found that both plane Poiseuille flow and plane Couette flow can be unstable, however small the Reynolds number is. The unstable modes are in the neighbourhood of a hidden neutral mode for the case of a single fluid, which is entirely ignored in the usual theory of hydrodynamic stability, and are brought out by the viscosity stratification.

1. Introduction

In this paper the stability of two superposed fluids of different viscosities in plane Couette and Poiseuille flow is considered. General formulas for calculating the eigenvalues of the complex wave velocity and thus for determining the stability or instability are given. Numerical calculations based on these formulas for plane Couette flow and plane Poiseuille flow for equal densities of the fluids have brought out the rather surprising result that the flow can be unstable at any Reynolds number, however small. Since for a single fluid plane Couette flow is known to be stable for all Reynolds numbers, however large, and plane Poiseuille flow is stable except at large Reynolds numbers, the instability mentioned above can only arise from the viscosity difference. The instability is even more striking if we consider first the plane Couette flow of a single fluid, which is stable. Upon *increasing* the viscosity of a layer of this flow, it becomes unstable.

The instability mentioned can already be inferred from the thesis of Sangster (1964). But his work was limited to an almost vertical flow. Therefore the instability he found is still mainly due to the longitudinal component of gravity. Only the increase of the degree of instability by viscosity variation found by him is pertinent to the present work, in which the body force has no longitudinal component.

2. The primary flow

Since it has been shown by Squire (1933) for channel flow of a uniform fluid between rigid boundaries, and later by Yih (1955) for stratified fluids, that it is sufficient to consider two-dimensional disturbances, we need only to write down the equations governing two-dimensional motion of viscous fluids. From these the primary flow can be readily determined. These equations are the Navier–Stokes equations

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial X} + \nu \Delta u, \tag{1}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial Y} - g + \nu \Delta v, \tag{2}$$

in which u and v are the velocity components in the directions of increasing X and Y, as shown in figure 1, t is the time, ρ the density, p the pressure, ν the kinematic viscosity (μ/ρ), Δ the Laplacian in X and Y, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial X} + v\frac{\partial}{\partial Y}.$$

The direction of increasing Y is the direction of the vertical, so that the X-direction is horizontal. This is to avoid having a longitudinal component of gravity, which has been known to be destabilizing (Benjamin 1957; Yih 1954, 1963; Kao 1965), and thus to focus the cause of any instability to be found on the viscosity variation.

The primary flow (figure 1) has only one velocity component $\bar{u}$, which is independent of t and X. (2) states that $\bar{p}+\rho g Y$ is independent of Y, and (1) states that $d\bar{p}/dX$ is independent of X. Hence

$$K = -d\bar{p}/dX, \tag{3}$$

in which K is a constant. (1) then can be written as

$$d^2\bar{u}/dY^2 = -K/\mu, \tag{4}$$

in which μ takes the value μ_1 for the upper fluid and μ_2 for the lower fluid. (4) is to be solved for each fluid, with the boundary conditions that $\bar{u}$ is equal to a specified U_0 on the upper boundary and zero on the lower boundary, and that $\bar{u}$ and the shear stress $\mu\, d\bar{u}/dY$ must be continuous at the interface.

If U_0 is not zero, the dimensionless mean velocities in the two layers are defined to be

$$U_1 = \bar{u}_1/U_0, \quad U_2 = \bar{u}_2/U_0. \tag{5}$$

In terms of the dimensionless co-ordinates

$$x = X/d_1, \quad y = Y/d_1, \tag{6}$$

the mean-velocity distributions are

$$U_1 = A_1 y^2 + a_1 y + b, \tag{7}$$

$$U_2 = A_2 y^2 + a_2 y + b, \tag{8}$$

in which

$$\left.\begin{aligned} A_2 &= -(\tfrac{1}{2}K/\mu_2 U_0)\, d_1^2, \quad A_1 = mA_2, \\ a_2 &= \{1 + A_2(n^2 - m)\}/(m+n), \quad a_1 = ma_2, \\ b &= \{1 - A_1(1+n)\}\, n/(m+n), \end{aligned}\right\} \tag{9}$$

with

$$m = \mu_2/\mu_1, \quad n = d_2/d_1. \tag{10}$$

If $\bar{u}(d_1) = 0$, it is convenient to define

$$U_1 = \bar{u}_1/\bar{u}(0), \quad U_2 = \bar{u}_2/\bar{u}(0). \tag{11}$$

The general expressions for U_1 and U_2 can be easily written. Since for $U_0 = 0$ we shall treat only the special case in which $\rho_1 = \rho_2$ and $d_1 = d_2$, these expressions will be given only for the case $n = 1$ and $r = \rho_2/\rho_1 = 1$:

$$U_1 = 1 + a_1 y + b_1 y^2, \tag{12}$$

$$U_2 = 1 + a_2 y + b_2 y^2, \tag{13}$$

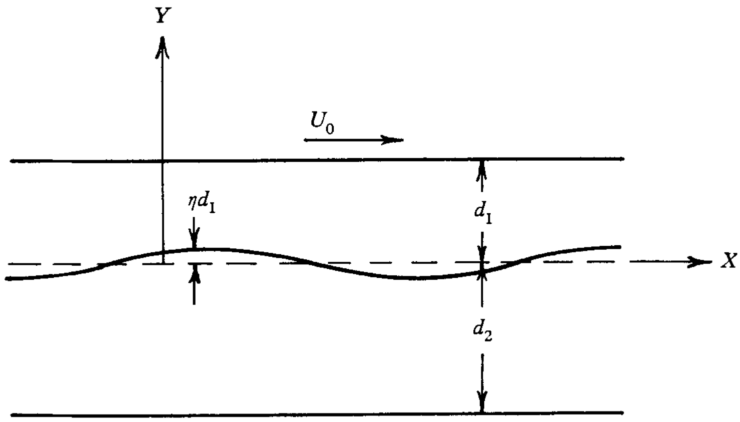

FIGURE 1. Definition sketch.

in which the subscripts 1 and 2 refer to the upper and lower fluids, respectively,

$$\left.\begin{aligned} a_1 &= \tfrac{1}{2}(m-1), & b_1 &= -\tfrac{1}{2}(m+1), \\ a_2 &= \tfrac{1}{2}(1-1/m), & b_2 &= -\tfrac{1}{2}(1+1/m). \end{aligned}\right\} \tag{14}$$

The reference velocity is the interfacial velocity, or $\overline{u}$ at $y = 0$. For the case $U_0 = 0$, $n = 1$ and $r = 1$, it is quite immaterial whether m is greater or less than 1. For definiteness we shall assume $m > 1$ in that case.

The velocity gradients at the interface are different for the two fluids if $m \neq 1$. This is what makes the instability considered here possible.

3. The differential system governing stability

It is well known that the differential equation governing stability is the Orr–Sommerfeld equation. In order to derive the normal-stress condition at the interface, one equation occurring in the derivations of the Orr–Sommerfeld equation is needed. For this reason, and for the sake of clarity and completeness, a brief derivation of that equation is included.

Apart from the equations of motion, the equation of continuity

$$\frac{\partial u}{\partial X} + \frac{\partial v}{\partial Y} = 0 \tag{15}$$

must be satisfied. We shall consider the upper layer first. With V denoting U_0 or $\overline{u}(0)$, as the case may be, the substitutions

$$\left.\begin{aligned} (\hat{u}, \hat{v}) &= (u, v)/V, \quad (x, y) = (X, Y)/d_1, \\ \hat{p} &= p/\rho_1 V^2 \quad \text{and} \quad \tau = tV/d_1 \end{aligned}\right\}$$

can be used to put (1), (2) and (15) in the dimensionless forms:

$$\frac{D\hat{u}}{D\tau} = -\frac{\partial\hat{p}}{\partial x}+\frac{1}{R}\Delta\hat{u}, \tag{1a}$$

$$\frac{D\hat{v}}{D\tau} = -\frac{\partial\hat{p}}{\partial y}+\frac{1}{R}\Delta\hat{v}, \tag{2a}$$

and

$$\frac{\partial\hat{u}}{\partial x}+\frac{\partial\hat{v}}{\partial y} = 0, \tag{15a}$$

in which R is the Reynolds number $\rho_1 Vd_1/\mu_1$, Δ now stands for the Laplacian

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},$$

and

$$\frac{D}{D\tau} = \frac{\partial}{\partial\tau}+\hat{u}\frac{\partial}{\partial x}+\hat{v}\frac{\partial}{\partial y}.$$

As usual, the motion is resolved into the primary motion and the perturbation motion. Thus

$$\hat{u} = U_1+u', \quad \hat{v} = v', \quad \hat{p} = P+p', \tag{16}$$

in which P is the dimensionless pressure for the primary flow. The equation of continuity is now in the form

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y} = 0,$$

which permits the use of the stream function ψ, in terms of which

$$u' = \psi_y, \quad v' = -\psi_x, \tag{17}$$

with the subscripts indicating partial differentiation. We shall now assume an exponential time factor for all perturbation quantities, and write

$$(\psi, p') = \{\phi(y), f(y)\}\exp i\alpha(x-c\tau), \tag{18}$$

in which $c = c_r+ic_i$. The stability or instability is then decided by the sign of c_i. If (16), (17), and (18) are substituted into (1 a) and (2 a), terms pertaining only to the primary flow are cancelled out, and quadratic terms in perturbation quantities are neglected, we have

$$i\alpha\{(U_1-c)\phi' - U_1'\phi\} = -i\alpha f+R^{-1}(\phi''' - \alpha^2\phi'), \tag{19}$$

$$\alpha^2(c-U_1)\phi = f' + (i\alpha/R)(\phi''-\alpha^2\phi), \tag{20}$$

in which primes on ϕ and U indicate differentiation with respect to y. Elimination of f from (19) and (20) produces the well-known Orr–Sommerfeld equation

$$\phi^{\text{iv}}-2\alpha^2\phi''+\alpha^4\phi = i\alpha R\{(U_1-c)(\phi''-\alpha^2\phi)-U_1''\phi\}, \tag{21}$$

which, together with the boundary conditions, governs the stability problem.

Equations (19), (20) and (21) are for the upper fluid. For the lower fluid, we choose to retain the substitutions (15) and the meanings of R and f, and to write χ for ϕ. (19) and (20) then become

$$i\alpha r\{(U_2-c)\chi' - U_2'\chi\} = -i\alpha f+(m/R)(\chi''' - \alpha^2\chi'), \tag{19a}$$

$$\alpha^2 r(c-U_2)\chi = f' + (i\alpha m/R)(\chi''-\alpha^2\chi), \tag{20a}$$

and (21) becomes

$$\chi^{\mathrm{iv}} - 2\alpha^2\chi'' + \alpha^4\chi = i\alpha R m^{-1} r\{(U_2 - c)(\chi'' - \alpha^2\chi) - U_2''\chi\}. \tag{21a}$$

The boundary conditions are

$$\phi(1) = 0, \quad \phi'(1) = 0, \tag{22}$$

$$\chi(-n) = 0, \quad \chi'(-n) = 0 \tag{23}$$

(expressing the condition of no slip at the rigid boundaries) and the interfacial conditions. The latter group consists of the conditions of continuity of velocity and of stresses. The continuity of v' (on which the accent does not indicate differentiation) demands

$$\phi(0) = \chi(0). \tag{24}$$

The continuity of u' must be formulated with more care, because the quantity U' is not continuous across the interface, and because the condition is to be imposed at $y = \eta$ (deviation of interface from its mean position) rather than at $y = 0$. Since

$$\left(\frac{\partial}{\partial \tau} + U\frac{\partial}{\partial x}\right)\eta = v' = -i\alpha\phi(0)\exp i\alpha(x - c\tau),$$

we have

$$\eta = \frac{\phi(0)}{c'}\exp i\alpha(x - c\tau), \quad c' = c - U(0). \tag{25}$$

The continuity in u' then demands

$$\phi'(0) + \frac{\phi(0)}{c'}U_1'(0) = \chi'(0) + \frac{\chi(0)}{c'}U_2'(0),$$

or

$$\phi'(0) - \chi'(0) = \frac{\phi(0)}{c'}(a_2 - a_1). \tag{26}$$

The continuity of shear stress is expressed by

$$\phi''(0) + \alpha^2\phi(0) = m\{\chi''(0) + \alpha^2\chi(0)\}. \tag{27}$$

Note that this boundary condition can be applied right at $y = 0$, for the gradient of shear stress is the same for both layers, so that the displacement of the surface can be ignored as far as the shear-stress condition at the interface is concerned.

The normal-stress condition at the interface is more complicated. The difference of the quantity

$$\rho g d_1 \eta - p'\rho_1 V^2 + \frac{2\mu V}{d_1}\frac{\partial v'}{\partial y} \tag{28}$$

evaluated for the upper fluid and that for the lower fluid must be

$$-\frac{T}{d_1}\frac{\partial^2\eta}{\partial x^2}. \tag{29}$$

In (28), the first term gives the negative of the hydrostatic pressure increment as y varies from zero to η, and the other terms are evaluated at $y = 0$, for either fluid. In (29) T denotes the surface tension. Expressed in dimensionless terms, the normal-stress condition is, upon utilization of (19) to evaluate f and hence p' for either fluid and of (25) to evaluate η,

$$-i\alpha R(c'\phi' + a_1\phi) - (\phi''' - \alpha^2\phi') + 2\alpha^2\phi' + ir\alpha R(c'\chi' + a_2\chi) + m(\chi''' - \alpha^2\chi') - 2\alpha^2 m\chi' = i\alpha R(F^{-2} + \alpha^2 S)\phi/c', \tag{30}$$

in which all variables are evaluated at $y = 0$, and

$$F^2 = \frac{\rho_2-\rho_1}{\rho_1}\frac{gd_1}{V^2}, \quad S = \frac{T}{\rho_1 d_1 V^2}, \quad c' = c - U(0). \tag{31}$$

(In any comparison of (30) with equation (25) in Yih (1963), the reader has to keep in mind the change of direction of the Y-axis.)

In the special case $\rho_1 = \rho_2$, or $r = 1$, (30) assumes the simpler form

$$m(\chi''' - 3\alpha^2\chi') - (\phi''' - 3\alpha^2\phi') = i\alpha^3 RS\phi/c' \quad \text{at} \quad y = 0. \tag{30a}$$

The differential system governing the stability problem consists of (21), (21a), (22), (23), (24), (26), (27) and (30). It defines an eigenvalue problem in the sense that given m, n, r, F, R and S, c has to take on certain values for the solution not to be identically zero. The flow is unstable, neutrally stable, or stable according as c_i is positive, zero, or negative.

4. Solution for the case of moving upper boundary

In this case $\bar{u} \neq 0$ at $Y = d_1$, and V is U_0. We shall adopt the method used by Yih (1963), which is essentially a method of non-singular perturbation around the case of $\alpha = 0$, which corresponds to very long waves. The quantity αR is assumed to be small compared with 1. Thus, however large R is, there is a small enough range of α for which the perturbation procedure is valid.

In the first approximation, all terms containing α in the differential system are ignored. In the second approximation, all terms containing α^2 and higher orders of α are ignored. Thus for the first approximation (21) and (21a) become

$$\phi_0^{\text{iv}} = 0 \quad \text{and} \quad \chi_0^{\text{iv}} = 0, \tag{32}$$

in which the subscripts zero indicate the first approximation. The boundary conditions (22), (23), (24), and (26) remain as they stand except that the variables all have subscripts zero, whereas (27) and (30) become

$$\phi_0''(0) - m\chi_0''(0) = 0, \tag{27a}$$

and

$$\phi_0'''(0) - m\chi_0'''(0) = 0. \tag{30b}$$

Solution of the differential system just formulated for $\alpha = 0$ gives

$$\left.\begin{aligned} \phi_0 &= 1 + B_1 y + C_1 y^2 + D_1 y^3, \\ \chi_0 &= 1 + B_2 y + C_2 y^2 + D_2 y^3, \end{aligned}\right\} \tag{33}$$

in which

$$B_1 = -\frac{m + 3n^2 + 4n^3}{2n^2(1+n)},$$

$$B_2 = \frac{2(m+n^3)}{mn} + \frac{n^2}{m}B_1,$$

$$C_1 = mC_2, \quad C_2 = \frac{m+n^3}{mn^2(1+n)},$$

$$D_1 = mD_2, \quad D_2 = \frac{n^2 - m}{2mn^2(1+n)}.$$

The eigenvalue c_0' is $c_0 - b$, and is determined by (26), which gives

$$c_0' = \frac{a_2 - a_1}{B_1 - B_2} = \frac{2mn^2(1+n)(a_1 - a_2)}{m^2 + 2mn(2+3n+2n^2)+n^4}. \tag{34}$$

When $m = 1$, a_1 is equal to a_2, and $c_0' = 0$. If $\phi_0(0)$ is not zero, the vanishing of c_0' would seem to make η infinite and present a difficulty. Actually if the magnitude of η is taken to be the standard, this situation merely means that when $c_0' = 0$ the velocity perturbations represented by $\phi(y)$ and $\phi'(y)$ are all zero, and only a corrugation remains. This point is intimately related to the difference in character of the unstable modes to be presented in this paper and the unstable modes treated in the usual theory of hydrodynamic stability.

The equations to be solved in the second approximation are

$$\phi_1^{\text{iv}} = i\alpha R\{(U_1 - c_0)\phi_0'' - 2A_1\phi_0\}, \tag{35}$$

and

$$\chi_1^{\text{iv}} = i\alpha Rm^{-1}r\{(U_2 - c_0)\chi_0'' - 2A_2\chi_0\}, \tag{36}$$

in which $2A_1$ has been written for U_1'' and $2A_2$ for U_2'', according to (7) and (8). The solution for (35) is

$$\phi_1 = \Delta B_1 y + \Delta C_1 y^2 + \Delta D_1 y^3 + i\alpha R h_1(y), \tag{37}$$

in which

$$h_1(y) = \frac{A_1 D_1}{210} y^7 + \frac{a_1 D_1}{60} y^6 + \frac{a_1 C_1 - 3c_0' D_1 - A_1 B_1}{60} y^5 - \frac{c_0' C_1 + A_1}{12} y^4. \tag{38}$$

The solution of (36) is

$$\chi_1 = \Delta B_2 y + \Delta C_2 y^2 + \Delta D_2 y^3 + i\alpha Rm^{-1}r h_2(y), \tag{39}$$

in which

$$h_2(y) = \frac{A_2 D_2}{210} y^7 + \frac{a_2 D_2}{60} y^6 + \frac{a_2 C_2 - 3c_0' D_2 - A_2 B_2}{60} y^5 - \frac{c_0' C_2 + A_2}{12} y^4. \tag{40}$$

In (37), the first three terms constitute the complementary solution necessitated by the last term, which is the particular solution. The term of zero degree in y is taken to be zero in (37). The argument is that the solution of the eigenvalue problem is determined only up to an arbitrary constant factor. We have taken the constant term of ϕ in (33) to be 1. We can and shall keep it at that value once and for all. That this will not deprive us of the possibility of satisfying the boundary conditions will presently be seen. For more detailed arguments, see Yih (1963, p. 326). Then the term of zero degree in y must also be zero in (39), as demanded by (24). The boundary conditions (22), (23), and (27) assume the forms

$$\Delta B_1 + \Delta C_1 + \Delta D_1 + i\alpha R h_1(1) = 0, \tag{22a}$$

$$\Delta B_1 + 2\Delta C_1 + 3\Delta D_1 + i\alpha R h_1'(1) = 0, \tag{22b}$$

$$-\Delta B_2 n + \Delta C_2 n^2 - \Delta D_2 n^3 + i\alpha Rm^{-1}r h_2(-n) = 0, \tag{23a}$$

$$\Delta B_2 - 2\Delta C_2 n + 3\Delta D_2 n^2 + i\alpha Rm^{-1}r h_2'(-n) = 0, \tag{23b}$$

$$m\Delta C_2 = \Delta C_1. \tag{27b}$$

We are then left with (26) and (30) to contend with. To the present order of approximation, (30) assumes the form

$$m\chi_1''' - \phi_1''' = i\alpha R\{(\phi_0/c_0' F^2) - r(c_0' \chi_0' + a_2 \chi_0) + (c_0' \phi_0' + a_1 \phi_0)\}, \tag{30c}$$

to be applied at $y = 0$. But (24) and (26), when applied to the first approximation, were

$$\phi_0'(0) - \chi_0'(0) = \{\phi_0(0)/c_0'\}(a_2 - a_1) \quad \text{and} \quad \phi_0(0) = \chi_0(0).$$

Hence (30c) can be written further as

$$m\chi_1''' - \phi_1''' = i\alpha R\{(\phi_0/c_0' F^2) - (r-1)(c_0' \phi_0' + a_1 \phi_0)\}, \tag{30d}$$

to be applied at $y = 0$. Thus

$$6m\Delta D_2 - 6\Delta D_1 = i\alpha R\{(1/c_0' F^2) - (r-1)(c_0' B_1 + a_1)\}. \tag{30e}$$

As to (26), its form for the second approximation takes some care, because c' also suffers a perturbation. With this in mind, and remembering that both $\phi_1(0)$ and $\chi_1(0)$ are zero, (26) becomes

$$\phi_1'(0) - \chi_1'(0) = -\{\Delta c \phi_0(0)/c_0'^2\}(a_2 - a_1),$$

in which Δc is the change in c and is of course identical with $\Delta c'$, since $U(0)$ does not change. Hence

$$(\Delta B_1 - \Delta B_2)\, c_0'^2 = -\Delta c(a_2 - a_1). \tag{26a}$$

The six Δ-coefficients can be found by solving (22a,b), (23a,b), (27b), and (30e). Then Δc can be found from (26a). The result is

$$\Delta c = ic_i, \quad c_i = \alpha R J(m, n, r, A_1), \tag{41}$$

in which

$$J = \frac{m^{-1} c_0'^2}{a_1 - a_2}\left\{m(h_1' - 2h_1) - J_2 - \frac{2}{n} H_2 + \frac{m - n^2}{2(1+n)}\left(h_1 - h_1' - \frac{J_2}{n} - \frac{H_2}{n^2}\right)\right\}, \tag{42}$$

with

$$h_1' = h_1'(1), \quad h_1 = h_1(1)$$

$$H_2 = rh_2(-n) - \tfrac{1}{6}n^3\{(1/c_0' F^2) - (r-1)(c_0' B_1 + a_1)\},$$

$$J_2 = rh_2'(-n) + \tfrac{1}{2}n^2\{(1/c_0' F^2) - (r-1)(c_0' B_1 + a_1)\}.$$

The method of regular perturbation adopted here has greatly reduced the algebraic work which otherwise would be necessary. But it is still desirable to provide an independent check of the correctness of (34), and the final results (41) and (42). For (34), the check is provided by the requirement that $c_0' = c_0 - b$ (b = mean velocity at the interface) must be equal in magnitude and opposite in sign if $r = 1$ and the depths and the viscosities of the layers are interchanged—or if m is replaced by m^{-1} and n by n^{-1}. Equation (34) withstands this test. For (41), the test is that the same interchanges should leave c_i (though not J) unchanged. (41) and (42) withstand this test. The author has checked these equations several times, and Mr Chin-Hsiu Li has checked them independently. They are free from errors. The numerical results obtained by the use of a computer also withstand the tests, indicating that the numerical results are also free from errors.

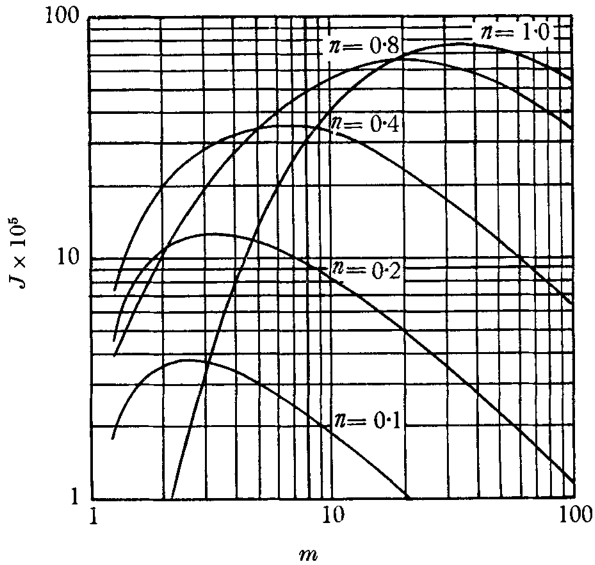

FIGURE 2 (*a*). Variation of J with the viscosity ratio m for various values of the depth ratio $n \leqslant 1$ for plane Couette flow with uniform density, showing instability.

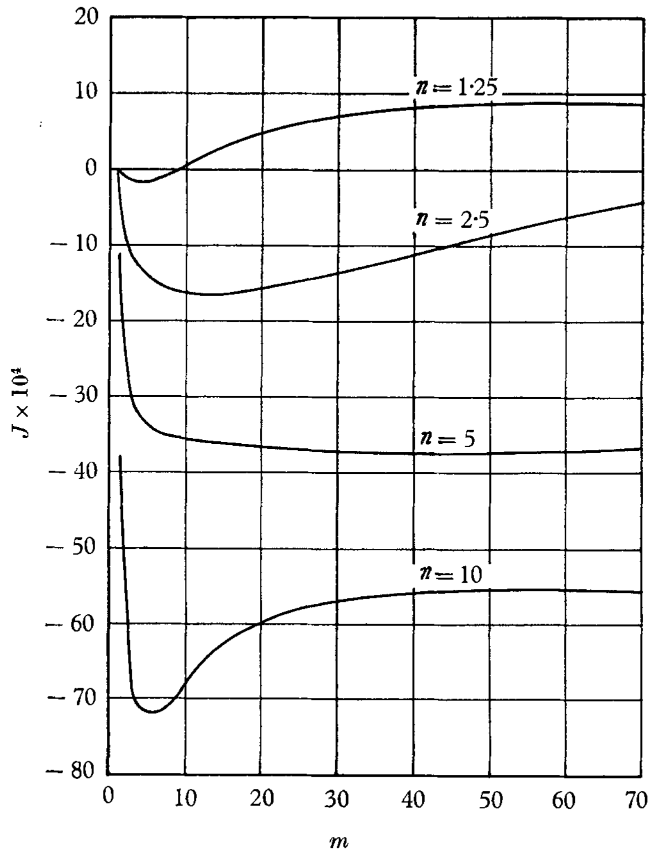

FIGURE 2 (*b*). Variation of J with the viscosity ratio m for various values of the depth ratio $n > 1$ for plane Couette flow with uniform density, showing extensive regions of instability.

From the definition of J given by (42) and for $r = 1$, the writer has verified that J vanishes as $(m-1)^2$ for $n = 1$ and as $m-1$ for $n \neq 1$, as m approaches 1. This point will be discussed further later.

To see whether the c_i calculated from (41) can be positive, numerical calculations have been carried out for the special case $A_1 = 0$ and $r = 1$. This is plane Couette flow, since there is no longitudinal pressure gradient in the mean flow. Since $r = 1$, there is no density difference, so that gravity has no effect on the phenomenon whatsoever, aside from imparting a hydrostatic part to the pressure. Keeping (41) in mind, one concludes from figure 2 (*a*) that for $n \leqslant 1$ the flow is unstable for all m greater than 1, with the instability greatest, for fixed αR, at some value of m between 2·5 (approximate) for $n = 0{\cdot}1$ and 35 for $n = 1$.

For $n > 1$ and $m > 1$ the variation of J with m and n is given in figure 2 (*b*). The value of J for $m = 1$ and different values of n is zero although the scale of figure 2 (*b*) does not allow two of the curves to be continued conveniently further to the left. Keeping this in mind, one can see from figure 2 (*b*) that for $n = 1{\cdot}25$ a region of stability exists between $m = 1$ and $m = 9$ approximately, and that for $n = 2{\cdot}5$, 5 and 10 the flow is stable up to $m = 70$ at least.

It can be directly verified from (42) that

$$\frac{m}{n^2} J(m, n) = J\left(\frac{1}{m}, \frac{1}{n}\right),$$

which corresponds to the statement that c_i is unchanged if the depths and viscosities of the layers are interchanged. This formula has been verified numerically. With this formula, the values of J for $m < 1$ for the n-values in figure 2 (*b*) can be obtained from the values of J given by the lower four curves in figure 2 (*a*) for $m > 1$, and vice versa.

Whenever the flow is unstable, the instability occurs at all Reynolds numbers, however small, although c_i does approach zero as R approaches zero. The instability is entirely due to viscosity variation. If $r > 1$, the gravity term in H_2 and J_2 will be stabilizing, and the destablizing effect of viscosity variation may be overshadowed.

5. Solution for the case of stationary boundaries

This is the case of plane Poiseuille flow. Since the general formulation has been amply illustrated in the preceding section, we shall further restrict ourselves to the case $r = 1$ and $n = 1$. Thus the two layers are of equal depth and have the same density. The mean velocity in the layers are given by (12) and (13). As mentioned in § 2, m can be assumed greater than 1 in the special case considered here.

Following the same approach as in § 4, we have

$$\phi_0 = 1 + B_1 y + C_1 y^2 + D_1 y^3, \tag{43}$$

of which the first term on the right-hand side has been assigned the value unity once and for all, and

$$\chi_0 = 1 + B_2 y + C_2 y^2 + D_2 y^3. \tag{44}$$

The first term in (44) has been determined by (24). The other boundary conditions determine the other coefficients to be

$$\left.\begin{aligned} B_1 = -\tfrac{1}{4}(7+m), \quad B_2 = \tfrac{1}{4}(1+7m)/m, \quad C_1 = \tfrac{1}{2}(1+m), \\ C_1 = mC_2, \quad D_1 = \tfrac{1}{4}(1-m), \quad D_1 = mD_2, \end{aligned}\right\} \tag{45}$$

and the eigenvalue to be

$$c_0 = 1+2(m-1)^2/(m^2+14m+1). \tag{46}$$

Before going to the second approximation, we shall pause to consider (46) and see whether the velocity of the primary flow is equal to c_0 at some point in the flow. Such a point has been called the critical point in the literature. It needs special attention if the viscous terms in the Orr–Sommerfeld equation are neglected at large Reynolds numbers to provide two (out of a total of four) asymptotic solutions, or if the diffusive terms in the linearized diffusion equation (if diffusion is part of the problem) are neglected. In the present problem we are not using any asymptotic solutions of the sort that require special treatment of the critical point. Hence the point at which $U = c$ is really not critical except perhaps at $y = 0$, where one boundary condition, (26), involves $c-U(0)$ or $c-1$. But since n is not equal to 1, (46) shows that c_0 is never equal to 1.

We now proceed to the second approximation, and to solve the equations

$$\phi_1^{\text{iv}} = i\alpha R\{(U_1-c_0)\,\phi_0'' - U_1''\phi_0\},$$

and

$$\chi_1^{\text{iv}} = i\alpha Rm^{-1}\{(U_2-c_0)\,\chi_0'' - U_2''\chi_0\}.$$

The solutions are

$$\phi_1 = \Delta B_1 y + \Delta C_1 y^2 + \Delta D_1 y^3 + i\alpha R h_1(y), \tag{47}$$

$$\chi_1 = \Delta B_2 y + \Delta C_2 y^2 + \Delta D_2 y^3 + i\alpha R m^{-1} h_2(y), \tag{48}$$

in which the variation of the first term in (44) is zero because of (24), and

$$h_1(y) = \frac{m^2-1}{1680}y^7 - \frac{(m-1)^2}{480}y^6 + \frac{m^4+18m^3-156m^2-98m-21}{480(m^2+14m+1)}y^5 - \frac{m^3-17m^2-17m+1}{24(m^2+14m+1)}y^4, \tag{49}$$

$$h_2(y) = \frac{m^2-1}{1680m^2}y^7 - \frac{(m-1)^2}{480m^2}y^6 + \frac{21m^4+98m^3+156m^2-18m-1}{480m^2(m^2+14m+1)}y^5 - \frac{m^3-17m^2-17m+1}{24m(m^2+14m+1)}y^4. \tag{50}$$

The boundary conditions lead, in a manner similar to that explained in § 4, to

$$\Delta c = ic_i, \quad c_i = 8\alpha R H_3, \tag{51}$$

in which

$$H_3 = \left(\frac{1-m}{m^2+14m+1}\right)^2 [-\tfrac{1}{2}(m+1)\{h_1(1)+h_2(-1)+h_2'(-1)-h_1'(1)\} - \tfrac{1}{4}(m-1)\{h_1(1)-h_1'(1)-h_2(-1)-h_2'(-1)\} - mh_1(1) - h_2(-1)]. \tag{52}$$

Equations (46) and (51) must pass the test that if the viscosities are interchanged, that is, if m is replaced by $1/m$, both c_0' $(=c_0-1)$ and c_i must remain unchanged. They do pass the test.

The function H_3 is plotted against m in figure 3, in which it can be seen that, for the case of equal depths and equal densities at least, plane Poiseuille flow is always unstable at any Reynolds number, however small. Note that since (52) indicates that H_3 vanishes as $(m-1)^2$ as m approaches zero, the curve in figure 3 dips asymptotically near the axis $m = 1$ as m approaches 1.

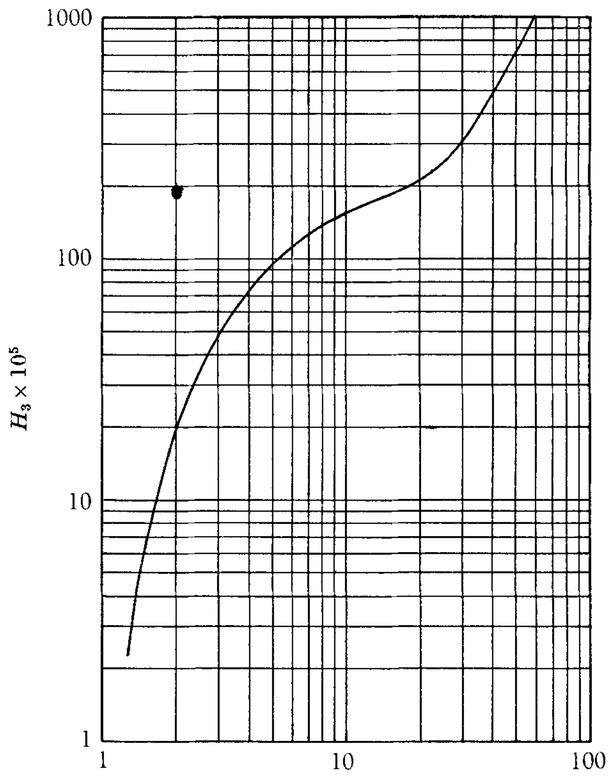

FIGURE 3. Curve showing instability of plane Poiseuille flow, $n = 1 = r$.

6. Discussion

The instability found in this paper can be regarded in two ways. In the first, the fluid is considered to be viscous to start with, and flows which are known to be stable for all Reynolds numbers or unstable only at high Reynolds numbers have been found to be unstable for any Reynolds number whatsoever if the viscosity varies from layer to layer and the depth ratio and the viscosity ratio are within certain ranges. In the second, the fluid can be regarded as inviscid and the velocity profile is assumed to have a discontinuity in slope. The stability of such flows have been studied by Rayleigh (1894, pp. 382–98) who found that the flows are stable when the slope of the velocity profile varies monotonically and unstable otherwise. The analogue of this conclusion when the velocity profile is continuous is well known. In the present work flows (such as plane Couette flow with a broken-line velocity profile) which are stable when the fluid is considered inviscid have been shown to be unstable when the viscosity is considered. From this point of view, the instability found here is due to the existence of viscosity. Of course, to obtain the velocity profiles treated in this work, the viscosity must vary from layer to layer.

What does finally become of the flow when it is unstable and disturbed slightly? Since the instability, when it exists, exists for any Reynolds number however small, one certainly does not expect turbulence to be the final result of instability when R is small. The long waves considered here will grow, but as soon as their amplitude becomes finite non-linear effects must be taken into account. Thus the present work can be considered to have demonstrated the possibility of finite waves in superposed layers of fluids of different viscosities.

But the results obtained still seem to run counter to intuition. To make them credible we shall show how they can be reconciled with the known results that plane Couette and plane Poiseuille flows for a single layer are either stable or at least stable for low Reynolds numbers. A single-fluid layer is obtained if the viscosity and density of one layer become the same as those of the other layer, or if the viscosity of one layer becomes infinite while that of the other layer remains finite. We shall discuss the two cases separately.

To see what happens if the density and viscosity of one layer become equal to those of the other, consider the results (41) and (51), both of which are for the case of equal density. Remembering (9), (34), and (42), we see after a brief calculation that both (41) and (51) give, for $n = 1$,

$$c_i = O\{(m-1)^2\}. \tag{53}$$

For $n \neq 1$ one can show from (41) that c_i vanishes as $m-1$ as m approaches 1. Thus c_i vanishes as $m-1$ or $(m-1)^2$ as m $(=\mu_2/\mu_1)$ approaches unity. Does this contradict the known results that there is no neutral mode for long waves at low Reynolds numbers? Superficially it seems to, but not in reality. Long waves in plane Couette or plane Poiseuille flow are damped according to long established results. But the mode considered in the investigations producing those results is quite different from the mode considered here. First of all, in those investigations αc is supposed finite. This already rules out finite c for the very long waves (vanishing α) considered here. For a detailed discussion, see Yih (1963, pp. 330–4). The mode considered in this paper reduces to a neutral mode for long waves as m approaches 1, but it is quite a different mode. Not only is c finite, but $U-c$ is zero at the imaginary interface—imaginary when m actually becomes unity. Since ϕ is not zero at $y = 0$ according to (33) and (43) for any m different from 1, and therefore is not zero in the limiting case of $m = 1$, and since we can take the magnitude of η as the standard magnitude of the perturbation, when $U-c$ vanishes at $y = 0$ in the limiting case of $m = 1$ the only conclusion to be drawn is that when m is equal or nearly equal to unity ϕ is generally an order of magnitude smaller than η, so that the mode is characterized by zero† perturbation in velocity and the existence of only a corrugation of the interface, in the limiting case, or by very small velocity perturbation compared with η, in case m is nearly equal to unity. The mode being discussed here for the limiting case of $m = 1$ is so drastically different from the damped long-wave modes in the usual theory, that

† After demanding that η be finite.

it may serve to dramatize the difference by calling it the 'soft' mode and the corresponding waves 'soft' waves as opposed to the Tollmien–Schlichting waves which may be considered 'hard'. In the neighbourhood of the 'soft' mode for $m = 1$, wherever the imaginary interface is, we have shown that there are unstable modes when m is different from 1. In other words the unstable modes discussed in this paper are not 'near' the damped modes considered in the usual theory, but near the soft modes ignored by it, and are brought out when there is a discontinuity in viscosity.

The arguments advanced here are quite similar to those I gave in an appendix to Benjamin's (1957) paper and again in my own (1963) paper to explain the instability of a liquid layer flowing down a vertical plane at very low Reynolds numbers, which at first also seemed unbelievable. Now it is generally accepted and it is understood that its cause is the longitudinal component of gravity, which supplies the power to the unstable disturbance. What supplies the power to the unstable disturbances treated in this paper is either the moving plate or the pressure gradient. Whether the disturbance can derive power from the mean flow sustained by these sources through the non-linear terms in the Navier–Stokes equations of motion can only be determined from a detailed study of the Orr–Sommerfeld equation and the boundary conditions. And such a study is precisely what has been done here. One can in fact obtain an integral formula from the Orr–Sommerfeld equation and the boundary conditions and interpret the formula from the point of view of energy. When the eigenfunction ϕ found in the solution is substituted in the integrals of that formula, c_i will be what we have given. That means unstable disturbances can be sustained by the available sources of power.

The limiting case of infinite viscosity in either layer will now be discussed. In that case only one layer is flowing, and the flow becomes an ordinary plane Couette–Poiseuille flow. The two special flows treated in numerical detail in this paper then become either an ordinary plane Couette flow (§ 4) or an ordinary plane Poiseuille flow. The former is known to be stable for all Reynolds numbers. The latter, as shown by Heisenberg (1924) and Lin (1945), is certainly stable at low Reynolds numbers. Do the results of these limiting cases contradict the present results? This is a question that is certain to occur in the minds of workers in hydrodynamic stability.

The answer is in the negative. The most important thing to recognize is that a rigid plane boundary is not a proper limit of the interface as μ_1 or μ_2 approaches infinity. If μ_1 (or μ_2) is large, the rate of deformation is very small in the very viscous layer. But given enough time the interface will deform and become wavy. Thus, however large μ_1 or μ_2 is, the interface should not be considered a rigid plane boundary. Thus the apparent contradiction does not really exist.

But it would be reassuring to see what happens to the analysis as μ_1 or μ_2 approaches infinity. It is sufficient to consider the case treated in § 4 ($U_0 \neq 0$) as μ_1 approaches infinity, or as $m \to 0$, because the other limiting cases behave similarly. In this case (9) shows that A_2, a_2, and b are all of order 1 as far as μ_1 or m is concerned. A_1 and a_1 are of the order of m. So is c_0', from (34). From the equations preceding (34) it is evident that B_1, C_1, and D_1 are all of order 1, whereas B_2, C_2, and D_2 are of order m^{-1}. This makes h_2, h_2' of order $1/m$, and hence J of order 1. Hence from (41) c_i is of the order of R, which is of the order of m (or μ_1^{-1}). Thus in the limit not only c_0' but also $\Delta c' = ic_i$ are both zero. Thus even to the second approximation c' is zero. From (25) it can be seen that η is of the order m^{-1} and therefore greater than $\phi(0)$ or v' by an order of magnitude. If the magnitude of η is taken to be the standard, then $\phi(0)$ or $\chi(0)$ should be zero. Why are they not zero in (33)? Remembering that the eigenfunctions of the differential system are determined only up to a multiplicative constant, we see that multiplication of (33) by m would make ϕ_0 equal to zero in the limit and, what is more revealing, would in the limit remove only the constant term from χ_0, since B_2, C_2, and D_2 are of order m^{-1}. This shows that $\chi_0(0)$ is indeed zero in the limit. Higher-order approximations do not change the fact that $\chi(0)$ is zero. Thus the apparent difficulty encountered in (25), (26), and (30) as $c' \to 0$ is resolved for the case $\mu_1 \to \infty$, and similarly for the case $\mu_2 \to 0$. The limiting flow corresponds in fact to the flow with a wavy interface, on which the velocity perturbations u' and v' are exactly zero. It is a case of neutral stability because it is simply the flow in a channel with a wavy wall. And we have demonstrated that in the neighbourhood of this seemingly insignificant case of neutral stability are innumerable cases of instability due to viscosity variation.

Finally, two points will be emphasized. First, the analysis presented here is fully applicable to two fluids of different densities as well as viscosities. Examples have been given for two fluids of equal density and different viscosities only to dramatize the fact that the instability is due to viscosity variation alone, and cannot be attributed to anything else. Secondly, since the fluids have been considered to be non-diffusive in viscosity as well as in density, the criticism might be raised that the instability discovered might be a result of neglecting to treat properly the so-called critical layer when the diffusion equation

$$D\mu/Dt = \kappa\Delta\mu$$

is truncated to

$$D\mu/Dt = 0$$

upon the neglect of diffusivity. There is no basis for this criticism. For in the results given here c is never equal to U at the interface, implying that, had the viscosity variation been continuous, the place where c is equal to U_1 or U_2 for the *corresponding* mode would fall outside of the range of viscosity variation. The question of the critical layer therefore does not arise. Note that the question concerning the ordinary critical layer also does not arise since Rayleigh's equation is never used instead of the Orr–Sommerfeld equation. The instability found here simply is unaffected by the non-consideration of the function of the critical layer in the analogous case of continuous viscosity variation. From the physical point of view non-diffusity is not unrealistic since there are many oils which do not mix at all with water, for instance.

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REFERENCES

BENJAMIN, T. B. 1957 Wave formation in laminar flow down an inclined plane. *J. Fluid Mech.* **2**, 554.

HEISENBERG, W. 1924 Über stabilität und Turbulenz von Flüssigkeitesströmen. *Ann. Phys. Lpz.* (4), **74**, 577–627

KAO, T. W. 1965 Stability of two-layer viscous stratified flow down an inclined plane. *Phys. Fluids* **8**, 812.

LIN, C. C. 1945 On the stability of two-dimensional parallel flows. Parts I, II, III. *Quart. Appl. Math.* **3**, 117–142, 218–234, 277–301.

LORD RAYLEIGH 1894 *Theory of Sound*, 2nd ed. London: MacMillan.

SANGSTER, W. M. 1964 The stability of stratified flows on nearly vertical slopes. Dissertation, State University of Iowa.

SQUIRE, H. B. 1933 On the stability for three-dimensional disturbances of viscous fluid flow between parallel walls. *Proc. Roy. Soc.* A **142**, 621.

YIH, C.-S. 1954 Stability of parallel laminar flow with a free surface. *Proc. 2nd U.S. Congr. of Appl. Mech.*, p. 623.

YIH, C.-S. 1955 Stability of two-dimensional parallel flows for three-dimensional disturbances. *Quart. Appl. Math.* **12**, 434.

YIH, C.-S. 1963 Stability of liquid flow down an inclined plane. *Phys. Fluids*, **6**, 321.